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Multi-resolution analysis and fractional quantum Hall effect: more results

F Bagarello

Dipartimento di Matematica ed Applicazioni, Fac. Ingegneria, Università di Palermo, Viale delle Scienze, I-90128 Palermo, Italy

E-mail: bagarell@unipa.it

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Abstract

In a previous paper we have proved that any multi-resolution analysis of $\mathcal{L}^2(\mathbb{R})$ produces, for even values of the inverse filling factor and for a square lattice, a single-electron wavefunction of the lowest Landau level (LLL) which, together with its (magnetic) translate, gives rise to an orthonormal set in the LLL. We have also discussed the inverse construction.

In this paper we simplify the procedure, clarifying the role of the *kq*-representation. Moreover, we extend our previous results to the more physically relevant case of a triangular lattice and to odd values of the inverse filling factor. We also comment on other possible shapes of the lattice as well as on the extension to other Landau levels.

Finally, just as a first application of our technique, we compute (an approximation of) the Coulomb energy for the Haar wavefunction, for a filling $\nu = \frac{1}{3}$.

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1. Introduction

Since its discovery, the fractional quantum Hall effect (FQHE) has been considered as a very exciting problem to be understood. For this reason, it is not surprising that the presence of the plateaux in the Hall resistivity has been explained in many different ways by many different people. Most of these proposals are analysed in monographs such as [20, 6, 10]. The current belief is that the Laughlin wavefunction Ψ_L can explain the existence of these plateaux. As is known, Ψ_L corresponds to an incompressible fluid and it describes electrons which are strongly correlated. It is also well known, however, that, for certain values of the filling factor ν, the electrons behave like a two-dimensional crystal, so that they must be described by a substantially different wavefunction, $\Psi^{(N)}$, which is usually taken as a normalized Slater determinant constructed from single electron normalized wavefunctions, which should be

mutually orthogonal in order $\Psi^{(N)}$ to be normalized in the thermodynamical limit [3]. Many proposals in this direction have been made over the years, [22, 16, 18, 23, 14, 9] among others, most of which are unable to reproduce the critical value of the filling factor for which a transition from liquid to crystal has been observed [17, 11, 26].

In this paper we are mainly interested in those values of ν for which the electrons behave as a two-dimensional crystal. This aspect of the FQHE has already been considered in many papers [22, 18, 14]. What is new here is the use of a technique coming from wavelet theory, the multi-resolution analysis (MRA) of $\mathcal{L}^2(\mathbb{R})$, which appears to be rather promising, as we will show in this work.

In a previous paper [2], we have proved that for a two-dimensional electron gas (2DEG) in a strong orthogonal magnetic field there exists a deep relation between any MRA of $\mathcal{L}^2(\mathbb{R})$ and a normalized wavefunction for the N electron system in the lowest Landau level (LLL). In particular, we have seen that any MRA produces a single electron wavefunction which, together with its (magnetic) translate, produces an orthonormal set in the LLL. We have also proved that the procedure can be inverted. This surprising result, however, was obtained by assuming that the electrons live on a square lattice and, more importantly, was shown to hold only for a filling factor $\nu = \frac{1}{2L}$, $L \in \mathbb{N}$. These mathematical constraints are physically rather unpleasant. In fact, classical and quantum arguments suggest that for low density the electrons should arrange themselves into a triangular lattice [5, 3], rather than into a square one. This is due to the fact that the triangular lattice minimizes, for a fixed electron density, the Coulomb energy. Therefore, in order to have a deeper understanding of the concrete relevance of wavelets in FQHE, it is absolutely necessary to extend the results in [2] to the triangular lattice. Moreover, the analysis of this new geometry gives a new insight into the procedure and, in particular, clarifies the role of the kq-representation, which was an essential mathematical tool in our original approach.

This is not the end of the story. It is well known that the main plateaux in the experimental measures of the resistivity tensor appear for filling factors such as $\nu = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$. This fact seems to suggest that our technique [2] is useless in most physically relevant situations (even if filling factors such as $1/2n$ have a physical interest in their own right!).

In this paper we overcome both these limitations: we first show how to obtain an orthonormality condition for the single electron wavefunctions for the triangular lattice. We also show that the same procedure can be adapted to lattices of arbitrary shape, provided that the so-called rationality condition is satisfied. Incidentally, this extension reduces the relevance of the kq-representation with respect to [2].

We also prove another result which looks quite fascinating to us: the odd values of ν^{-1} are recovered by MRA of a different kind, that is MRA with dilation parameter d different from 2. In particular, we will prove that a d-MRA, d natural and larger than 2, produces an orthonormal set of single electron wavefunctions in the LLL related to a filling factor $\nu = \frac{1}{d}$. The extension to other Landau levels, as well as the possibility of reversing the construction, is also discussed in some detail.

The paper ends with an example of a 3-MRA which extends the usual Haar construction [7], which is used to produce an N-electron Slater determinant made up of orthonormal single electron wavefunctions, for $\nu = \frac{1}{3}$. Finally, we compute the quantum Coulomb energy associated with this state under some simplifying approximations.

2. Mathematical tools

In order to keep the paper self-contained, we now quickly review, for the reader's convenience, the main definitions of the mathematical tools we will use in the rest of the paper.

2.1. *d-multi-resolution analysis*

The main result in the theory of d-MRA is the recipe which allows us to construct an orthonormal basis in $\mathcal{L}^2(\mathbb{R})$ starting from a set of functions $\psi^{(i)}$, $i = 1, 2, \ldots, d-1$, and acting on these with dilation and translation operators.

The full story for $d = 2$ may be found, for instance, in [7], which also contains some remarks concerning $d = 3$. For a complete reading on this subject we suggest [21, 13], where the definition of MRA is extended to $\mathcal{L}^2(\mathbb{R}^n)$ and the dilations and translations are not necessarily related to the integer numbers. Here, however, we give the definition of a d-MRA for $\mathcal{L}^2(\mathbb{R})$ and d integer (and $d \geqslant 2$), and then we focus only on those aspects which will be useful in the following.

A *d-multi-resolution analysis* of $L^2(\mathbb{R})$ is an increasing sequence of closed subspaces

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \tag{2.1}$$

with $\bigcup_{j\in\mathbb{Z}} V_j$ dense in $\mathcal{L}^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, and such that

(1) $f(x) \in V_j \Leftrightarrow f(\mathrm{d}x) \in V_{j+1}$
(2) There exists a function $\phi \in V_0$, called a *scaling* function, such that $\{\phi(x-k), k \in \mathbb{Z}\}$ is an o.n. basis of V_0.

Combining (1) and (2), one gets an o.n. basis of V_j, namely $\{\phi_{j,k}(x) \equiv d^{j/2}\phi(d^j x - k), k \in \mathbb{Z}\}$.

For an integer $d \geqslant 2$ the existence of a set of functions $\psi^{(i)}$, $i = 1, 2, \ldots, d-1$, such that $\left\{\left\{\psi^{(i)}_{j,k}(x) \equiv d^{j/2}\psi^{(i)}(d^j x - k),\ j, k \in \mathbb{Z}\right\},\ i = 1, 2, \ldots, d-1\right\}$ constitutes an o.n. basis of $\mathcal{L}^2(\mathbb{R})$ can be proved. We are not interested in all the details of this construction here, see [21, 13], but only for some easy consequences of this definition.

Note that the inclusion $V_0 \subset V_1$ yields the relation

$$\phi(x) = \sqrt{d}\sum_{n=-\infty}^{\infty} h_n \phi(\mathrm{d}x - n) \qquad h_n = \langle\phi_{1,n}|\phi\rangle = \sqrt{d}\int_{\mathbb{R}} \overline{\phi(\mathrm{d}x - n)}\phi(x)\,\mathrm{d}x. \tag{2.2}$$

The orthonormality requirement of the functions ϕ, $\int_{\mathbb{R}} \overline{\phi(x-n)}\phi(x)\,\mathrm{d}x = \delta_{n,0}$, together with equation (2.2), produces the key equation for the coefficients h_n:

$$\sum_{n\in\mathbb{Z}} h_n \overline{h_{n+dl}} = \delta_{l,0} \tag{2.3}$$

which generalizes equation (2.12) of [2] to a dilation parameter $d \geqslant 2$.

It is also easily checked that, if $\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x \neq 0$, then the coefficients must also satisfy the sum rule

$$\sum_{n\in\mathbb{Z}} h_n = \sqrt{d}. \tag{2.4}$$

Equation (2.3) is all we will need in the following. It is clear from the above construction that any d-MRA produces a set of complex quantities h_n which satisfies this condition. The converse procedure, which is not so interesting for our aims, is discussed in [21] and requires some care: in particular, it is easy to produce examples, for $d \geqslant 3$, of $\{h_n\}$ satisfying condition (2.3) and a *scaling function* ϕ satisfying the d-scale relation (2.2), which do not produce a d-MRA.

2.2. *kq-representation*

We give here only a few definitions, namely those relevant for this paper, and refer to [24, 15, 25] for further reading and for applications.

The genesis of the kq-representation lies in the well-known possibility of a simultaneous diagonalization of any two commuting operators. In particular, the following distributions:

$$\psi_{kq}(x) = \sqrt{\frac{2\pi}{a}} \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}kna}\delta(x-q-na) \qquad k \in [0, a[\quad q \in \left[0, \frac{2\pi}{a}\right[\tag{2.5}$$

are (generalized) eigenstates of both $T(a) = \mathrm{e}^{\mathrm{i}pa}$ and $\tau\left(\frac{2\pi}{a}\right) = \mathrm{e}^{\mathrm{i}x2\pi/a}$. Here a is a given positive real number.

These $\psi_{kq}(x)$ are Bloch-like functions corresponding to infinitely localized Wannier functions [25]. They also satisfy orthogonality and closure properties. This implies that, roughly speaking, they can be used to define a new representation of the wavefunctions by means of the integral transform $Z : \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathcal{D}), \mathcal{D} = [0, a[\times \left[0, \frac{2\pi}{a}\right[$, defined as follows:

$$h(k,q) := (ZH)(k,q) := \int_{\mathbb{R}} \mathrm{d}\omega\, \overline{\psi_{kq}(\omega)} H(\omega) \tag{2.6}$$

for any given function $H(\omega) \in \mathcal{L}^2(\mathbb{R})$. The result is a function $h(k,q) \in \mathcal{L}^2(\mathcal{D})$.

To be more rigorous, Z should be defined first on the functions of $\mathcal{C}_o^\infty(\mathbb{R})$ and then extended to $\mathcal{L}^2(\mathbb{R})$ using the continuity of the map [8].

Replacing $\psi_{kq}(x)$ with its explicit expression, formula (2.6) produces

$$h(k,q) = (ZH)(k,q) = \sqrt{\frac{2\pi}{a}} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}kna} H(q+na) \tag{2.7}$$

which can be inverted and gives the x-representation $H(\omega) \in \mathcal{L}^2(\mathbb{R})$ of a function $h(k,q) \in \mathcal{L}^2(\mathcal{D})$ as follows:

$$H(\omega) = (Z^{-1}h)(\omega) = \int_{\mathcal{D}} \mathrm{d}k\, \mathrm{d}q\, \psi_{kq}(\omega) h(k,q). \tag{2.8}$$

Due to (2.5), this equation can be written as

$$H(\omega+na) = \sqrt{\frac{2\pi}{a}} \int_0^a \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kna} h(k,\omega) \qquad \forall\omega \in [0, a[\quad \forall n \in \mathbb{Z}. \tag{2.9}$$

3. Stating the problem

The physical system we are considering is a 2DEG, that is, a gas of electrons constrained in a two-dimensional layer belonging to the $(O; x, y)$ plane, in a neutralizing positive uniform background and subjected to a uniform magnetic field along z.

The Hamiltonian of the N-electron system can be written as

$$H^{(N)} = H_0^{(N)} + \lambda\left(H_c^{(N)} + H_B^{(N)}\right) \tag{3.1}$$

where $H_0^{(N)}$ is the sum of N contributions:

$$H_0^{(N)} = \sum_{i=1}^{N} H_0(i). \tag{3.2}$$

Here $H_0(i)$ describes the minimal coupling of the electrons with the magnetic field:

$$H_0 = \frac{1}{2}\left(\underline{p} + \underline{A}(r)\right)^2 = \frac{1}{2}\left(p_x - \frac{y}{2}\right)^2 + \frac{1}{2}\left(p_y + \frac{x}{2}\right)^2. \tag{3.3}$$

Note that we are adopting here the symmetric gauge $\underline{A} = \frac{1}{2}(-y, x, 0)$ and the same units as in [3]. $H_c^{(N)}$ is the canonical Coulomb interaction between charged particles:

$$H_c^{(N)} = \frac{1}{2}\sum_{i\neq j}^{N} \frac{1}{|\underline{r}_i - \underline{r}_j|} \tag{3.4}$$

and $H_B^{(N)}$ is the interaction of the charges with the background, whose explicit form is irrelevant here and can be found in [3].

In the following we will consider, as is usually done in the literature, $\lambda\left(H_c^{(N)} + H_B^{(N)}\right)$ as a perturbation of the free Hamiltonian $H_0^{(N)}$, and we will look for eigenstates of $H_0^{(N)}$ in the form of Slater determinants built up with single electron wavefunctions. This approach is known to give good results for low electron (or hole) densities, [18, 3]. However, comparison with the experiments also shows that the wavefunctions proposed in [3, 18, 22, 9] do not reproduce the experimental data completely, as far as the transition from the Wigner crystal to the Laughlin liquid is concerned. Other wavefunctions proposed in the literature share the same problem, so that the problem of finding the correct single-electron wavefunction describing the system for low electron density is still open. In this paper we will show how an o.n. set of wavefunctions in the LLL (as well as in any other Landau level) can be constructed easily starting from a d-MRA.

The easiest way to attack this problem consists in introducing the new variables

$$P' = p_x - y/2 \qquad Q' = p_y + x/2. \tag{3.5}$$

In terms of P' and Q' the single electron Hamiltonian, H_0, can be written as

$$H_0 = \tfrac{1}{2}(Q'^2 + P'^2). \tag{3.6}$$

The transformation (3.5) can be seen as part of a canonical map U from (x, y, p_x, p_y) into (Q, P, Q', P') where

$$P = p_y - x/2 + \tfrac{1}{\sqrt{3}}(p_x + y/2) \qquad Q = p_x + y/2. \tag{3.7}$$

These operators satisfy the following commutation relations:

$$[Q, P] = [Q', P'] = \mathrm{i} \qquad [Q, P'] = [Q', P] = [Q, Q'] = [P, P'] = 0. \tag{3.8}$$

Using [19], we deduced in [4] that, as a consequence of the above canonical transformation, a wavefunction in the (x, y)-space is related to its expression in terms of the new variables (P, P') by the formula

$$\psi(x, y) = \frac{\mathrm{e}^{\frac{\mathrm{i}y}{2}(x - \frac{y}{\sqrt{3}})}}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\{P'(x - \frac{y}{\sqrt{3}}) + yP + PP' - \frac{P'^2}{2\sqrt{3}}\}} \psi(P, P')\,\mathrm{d}P\,\mathrm{d}P'. \tag{3.9}$$

We want to stress that this formula differs from the analogous one in [2] since the map $U : (x, y, p_x, p_y) \to (Q, P, Q', P')$ differs from the one used in [2]. The reason for this difference is in the shape of the lattice, which produces a different canonical transformation. This will appear clearly in the following, when we will discuss how the lattice should be constructed explicitly and the role of the magnetic translations in this construction.

The usefulness of the new variables stems from expression (3.6) of H_0. Indeed, in this representation, the single electron Schrödinger equation admits eigenvectors $\psi(P, P')$ of H_0 of the form $\psi(P, P') = f(P')h(P)$. Thus the ground state of (3.6) must have the form $f_0(P')h(P)$, where

$$f_0(P') = \pi^{-1/4}\,\mathrm{e}^{-P'^2/2} \tag{3.10}$$

and the function $h(P)$ is arbitrary in $\mathcal{L}^2(\mathbb{R})$, which manifests the degeneracy of the LLL. With f_0 as above, formula (3.9) becomes

$$\psi(x, y) = \int_{-\infty}^{\infty} K_0^{(t)}(\underline{r}, s) h(s)\, \mathrm{d}s \tag{3.11}$$

where $K_0^{(t)}$ is what we call the kernel of the transformation,

$$K_0^{(t)}(\underline{r}, s) = \frac{\mathrm{e}^{\frac{\mathrm{i}y}{2}(x-\frac{y}{\sqrt{3}})}}{\pi^{3/4}\sqrt{2\left(1+\frac{\mathrm{i}}{\sqrt{3}}\right)}}\, \mathrm{e}^{\mathrm{i}ys-\beta(x-\frac{y}{\sqrt{3}}+s)^2} \tag{3.12}$$

where $\beta = \frac{3}{8}\left(1-\frac{\mathrm{i}}{\sqrt{3}}\right)$. Here we use the suffix t to emphasize the shape of the lattice, triangular in this paper, while the index 0 means that we are working in the LLL. Just to compare this result with the one obtained for the square lattice [2] we recall that $K_0^{(s)}(\underline{r}, s) = \frac{\mathrm{e}^{\mathrm{i}xy/2}}{\sqrt{2}\pi^{3/4}}\, \mathrm{e}^{\mathrm{i}ys}\, \mathrm{e}^{-(x+s)^2/2}$. The extensions to higher Landau levels (see again [2] for $K_1^{(s)}$) can be found by simply replacing f_0 in (3.10) with the excited harmonic oscillator eigenstates: using f_l will produce, clearly, a wavefunction in the lth Landau level. However, for large magnetic fields, it is well known that only the very first levels are relevant in the analysis of the FQHE, and this is the main reason why, quite often, only the LLL is considered. This is also our choice here, and for this reason, throughout this paper, we will consider only $K_0^{(t)}$, but for few comments.

Once the generic form of any function in the LLL has been given, formula (3.11), it remains to properly choose one of them, that is, to fix $h(s)$ by requiring that it minimizes the Coulomb interaction. To achieve this aim we will first construct a particular ground state of the (infinitely degenerate) free N-electron Hamiltonian $H_0^{(N)}$. We use a suggestion coming from the classical counterpart of this quantum problem. It is very well known that the ground state for a classical 2DEG is a (triangular) Wigner crystal: the classical electrons are sharply localized on the sites of a lattice whose lattice spacing is fixed by the electron density [5]. What we expect, and what was proved in [3], is that, at least for certain values of the filling factor, the quantum ground state should not be very different from this classical picture.

Let us introduce the so-called magnetic translation operators $T(\vec{a}_i)$ defined by

$$T(\vec{a}_i) \equiv \exp(\mathrm{i}\vec{\Pi}_c \cdot \vec{a}_i) \qquad i = 1, 2 \tag{3.13}$$

where $\vec{\Pi}_c \equiv (Q, P)$ and $\vec{a}_i$ are the lattice basis vectors:

$$\vec{a}_1 = a(1, 0) \qquad \vec{a}_2 = \frac{a}{2}(1, \sqrt{3}). \tag{3.14}$$

We assume the following (minimal) *rationality condition* on the area of the fundamental cell of the lattice:

$$a_{1_x} a_{2_y} - a_{1_y} a_{2_x} = 2\pi \tag{3.15}$$

which for the triangular lattice reads

$$a^2 = \frac{4\pi}{\sqrt{3}}. \tag{3.16}$$

Note that this is different from the analogous condition for a square lattice, $a^2 = 2\pi$ [2]. Because of this condition we have

$$[T(\vec{a}_1), T(\vec{a}_2)] = 0 \tag{3.17}$$

which, together with

$$[T(\vec{a}_1), H_0] = [T(\vec{a}_2), H_0] = 0 \tag{3.18}$$

which easily follows from (3.8), show that $T(\vec{a}_1)$, $T(\vec{a}_2)$ and H_0 are a set of commuting operators.

This implies that if ψ is an eigenstate of H_0 with eigenvalue ϵ, then $T(\vec{a}_1)^n T(\vec{a}_2)^m \psi$ is still an eigenstate of H_0 corresponding to the same eigenvalue, for all $n, m \in \mathbb{Z}$. This can be seen as further evidence of the infinite degeneracy of the different Landau levels.

Because of (3.14), the magnetic translations take the form

$$\begin{aligned} T_1 &:= T(\vec{a}_1) = \mathrm{e}^{\mathrm{i}a(p_x+\frac{y}{2})} = \mathrm{e}^{\mathrm{i}aQ} \\ T_2 &:= T(\vec{a}_2) = \mathrm{e}^{\mathrm{i}\frac{a}{2}((p_x+y/2)+\sqrt{3}(p_y-x/2))} = \mathrm{e}^{\mathrm{i}\frac{a\sqrt{3}}{2}P} = \mathrm{e}^{\mathrm{i}\frac{2\pi}{a}P} \end{aligned} \tag{3.19}$$

which show the relevance of the definitions in (3.7). Note that the last equality for T_2 above is a simple consequence of the rationality condition (3.16).

Using this equality we can check that the action of T_1 and T_2 on a generic function $f(x, y) \in \mathcal{L}^2(\mathbb{R}^2)$ is given by

$$f_{n,m}(x, y) := T_1^n T_2^m f(x, y) = (-1)^{mn}\, \mathrm{e}^{\frac{\mathrm{i}}{2}(Y_{nm}x - X_{nm}y)} f(x - X_{nm}, y - Y_{nm}). \tag{3.20}$$

We see from this formula that if, for instance, $f(x, y)$ is localized around the origin, then $f_{n,m}(x, y)$ is localized around the lattice site (X_{nm}, Y_{nm}). Here $X_{nm} = -a\left(n + \frac{m}{2}\right)$ and $Y_{nm} = -am\frac{\sqrt{3}}{2} = -m\frac{2\pi}{a}$. This result extends, as expected, the one for the square lattice [2] and explains why we call T_j *translation operators*.

Moreover, equation (3.20) is the key relation by means of which the quantum triangular lattice can be constructed: we simply start from the single electron ground state of H_0 given in (3.11), $\psi(x, y)$. Then we construct a set of copies $\psi_{m,n}(x, y)$ of ψ as in (3.20), with $m, n \in \mathbb{Z}$. All these functions still belong to the lowest Landau level for any choice of the function $h(P)$, due to (3.18), and are localized around the sites of our triangular lattice. N of these wavefunctions $\psi_{m,n}(x, y)$ are finally used to construct a Slater determinant for the finite system:

$$\psi^{(N)}(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{m_1,n_1}(\underline{r}_1) & \psi_{m_1,n_1}(\underline{r}_2) & \cdot & \cdot & \cdot & \cdot & \psi_{m_1,n_1}(\underline{r}_N) \\ \psi_{m_2,n_2}(\underline{r}_1) & \psi_{m_2,n_2}(\underline{r}_2) & \cdot & \cdot & \cdot & \cdot & \psi_{m_2,n_2}(\underline{r}_N) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \psi_{m_N,n_N}(\underline{r}_1) & \psi_{m_N,n_N}(\underline{r}_2) & \cdot & \cdot & \cdot & \cdot & \psi_{m_N,n_N}(\underline{r}_N) \end{vmatrix}. \tag{3.21}$$

We have already discussed in [2] the relevance of the orthonormality of differently localized single electron wavefunctions when the thermodynamical limit of the model is considered. For this reason we look for o.n. single electron wavefunctions. Therefore, let $\psi(x, y)$ be as in (3.11) and $\psi_{n,m}(x, y) = T_1^n T_2^m \psi(x, y)$. We are interested in finding conditions on $h(P)$ such that the orthonormality condition (ONC)

$$S_{n,m} := \int_{\mathbb{R}^2} \overline{\psi_{0,0}(x, y)}\psi_{n,m}(x, y)\, \mathrm{d}x\, \mathrm{d}y = \delta_{m,0}\delta_{n,0} \tag{3.22}$$

is satisfied. Any function $\psi(x, y)$ which satisfies this condition and which belongs to the LLL can be used to generate the function $\psi^{(N)}(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N)$ as in (3.21). Moreover, it is not hard to prove that the ONC can be rewritten as

$$S_{n,m} := \int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{\mathrm{i}sm\frac{2\pi}{a}} \overline{h(s)} h(s - na) = \delta_{m,0}\delta_{n,0}. \tag{3.23}$$

This is a consequence of equations (3.11), (3.12), (3.16) and (3.20).

It is interesting to remark that this equation implies that $\psi_{0,0}$ is normalized in $\mathcal{L}^2(\mathbb{R}^2)$ if and only if $h(s)$ is normalized in $\mathcal{L}^2(\mathbb{R})$. This is clearly a consequence of the canonicity of the map U.

Moreover, equation (3.23) coincides exactly with the analogous one we have obtained for the square lattice, under the condition $a^2 = 2\pi$ [2]. Incidentally, this fact strongly suggests that the same condition will appear independent of the shape of the lattice. Finally, we observe that the ONC (3.22) produces the same equation (3.23) independent of the particular Landau level we are considering. In other words, it is not hard to check that we obtain equation (3.23) even if the kernel $K_0^{(t)}$ is replaced by $K_l^{(t)}$, for any given $l \in \mathbb{N}$. This proves, therefore, that the problem of the orthonormality of the single electron wavefunctions is *Landau-level independent*. Of course, this does not mean that the o.n. wavefunctions look the same in any Landau level: this is true only at the level of the function $h(s)$. In contrast, the wavefunctions in the coordinate representation will be different because the kernels of the transformations, $K_l^{(t)}(\underline{r}, s)$, are different for different values of l!

If we require equation (3.23) to hold for all integers n, m we are implicitly assuming that all the lattice sites are occupied by an electron, so that the filling is $\nu = 1$.

In [2] we have discussed how the ONC should be rewritten in order to cover a fractional value of the filling factor. A possible way of requiring orthonormality between single electron wavefunctions for $\nu = \frac{1}{d}$, $d \in \mathbb{N}$, consists in replacing (3.23) with the following equation:

$$\int_{\mathbb{R}} \mathrm{d}s\, \mathrm{e}^{\mathrm{i}sdm\frac{2\pi}{a}} \overline{h(s)} h(s - na) = \delta_{m,0}\delta_{n,0} \tag{3.24}$$

for all $m, n \in \mathbb{Z}$, which is obtained under the assumption that only one lattice site every d is occupied. We see, therefore, that the ONC becomes very much *filling-dependent*, and, in a certain sense, this fact is not very surprising to us: a filling $\nu < 1$ can be interpreted as if in the physical system we have fewer electrons than for $\nu = 1$, so that these electrons are more distant from one another in order to minimize the Coulomb energy. For this reason we expect different analytic expressions for the single electron wavefunctions depending on the value of the filling.

We are now left with the main problem: is it possible to produce solutions of equation (3.24)? This problem has been partially solved in [2] where solutions were explicitly built up for even values of the inverse filling factor starting from a 2-MRA and making use of the kq-representation. In the next section we will show how to solve (3.24) both for odd and even values of ν^{-1}, and this will be done without even mentioning the kq-representation. However, in section 5, in order to show the relation to our previous approach, we will rewrite everything using the Zak transform.

4. *d*-MRA and odd inverse filling

In this section we propose a method for constructing solutions of equation (3.24) which works for all integer values of the inverse filling. This method produces exactly the same solutions as in [2] only for $\nu = \frac{1}{2}$, while it produces different wavefunctions for other even values of ν^{-1}. Moreover, it will produce solutions also for those values of ν^{-1} which were not covered by the approach in [2], that is for odd values of ν^{-1}.

Let us consider a given d-MRA of $\mathcal{L}^2(\mathbb{R})$ and its related square-summable set of complex numbers $\{h_n\}_{n\in\mathbb{Z}}$ satisfying condition (2.3). Following [2], we use these coefficients to define a function $T_d(\omega)$ as follows:

$$T_d(s) = \begin{cases} \frac{1}{\sqrt{a}} \sum_{l\in\mathbb{Z}} h_l \, \mathrm{e}^{\mathrm{i}ls\frac{2\pi}{a}} & s \in [0, a[\\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

It is clear that $T_d(s)$ is square integrable and not periodic. In particular, due to condition (2.3), we have $\|T_d\|_2^2 = \int_{\mathbb{R}} |T_d(s)|^2 \, \mathrm{d}s = 1$.

With this definition it is straightforward to check that

$$\int_{\mathbb{R}} \mathrm{d}s \, \mathrm{e}^{\mathrm{i}sdm\frac{2\pi}{a}} \overline{T_d(s)} T_d(s - na) = \delta_{n0}\delta_{m0}. \tag{4.2}$$

In fact:

(a) the supports of $T_d(s)$ and $T_d(s - na)$ are disjoint for $n \neq 0$;
(b)

$$\int_{\mathbb{R}} \mathrm{d}s \, \mathrm{e}^{\mathrm{i}sdm\frac{2\pi}{a}} |T_d(s)|^2 = \frac{1}{a} \sum_{l,p\in\mathbb{Z}} \overline{h}_l h_p \int_0^a \mathrm{d}s \, \mathrm{e}^{\mathrm{i}s\frac{2\pi}{a}(dm-l+p)} = \sum_{p\in\mathbb{Z}} h_p \overline{h}_{p+dm} = \delta_{m0}$$

due to condition (2.3).

The physical consequences of this fact are evident: calling

$$\psi_d(x, y) = \int_{\mathbb{R}} K_0^{(t)}(\underline{r}, s) T_d(s) \, \mathrm{d}s \tag{4.3}$$

the following ONC

$$\left\langle \psi_d, T_1^n T_2^{dm} \psi_d \right\rangle = \delta_{n0}\delta_{m0} \qquad \forall n, m \in \mathbb{Z} \tag{4.4}$$

holds. We must stress that our construction does not imply that ψ_d is orthogonal to $T_1^i T_2^j \psi_d$ for all integer values of i and j, but only for those indices (i, j) which belong to the set $\{(\mathbb{Z}, d\mathbb{Z})\}$.

The set $\left\{T_1^n T_2^{dm} \psi_d(x, y) : n, m \in \mathbb{Z}\right\}$ generates a triangular lattice where not all the sites are occupied: with our choice, only a site every d along the $\vec{a}_2$ direction is nonempty, while all the sites along $\vec{a}_1$ are full. It is evident that this is only one among many choices: another possibility is exactly the opposite one: all the sites are occupied along $\vec{a}_2$ while only one in every d is nonempty along $\vec{a}_1$. This choice corresponds to the set $\left\{T_1^{dn} T_2^m \psi_d(x, y) : n, m \in \mathbb{Z}\right\}$. More symmetrical choices are not difficult to imagine. Clearly, from a mathematical point of view, all these choices are equivalent. However, physical reasons suggest that numerical differences could arise in the computation of the Coulomb energy, the reason being the existence of possible different asymptotic behaviour of the single electron wavefunctions for different choices of the above sub-lattices. An '*a priori*' decision seems very hard: the only reasonable criteria we could use are suggested by symmetry considerations. However, we will see in section 6 that already the choice discussed above, even if it breaks down the symmetry for rotations of $\pi/3$ which one would expect for the triangular Wigner crystal, produces a fairly good localized eigenfunction of H_0 and, therefore, is surely worth a deeper analysis also from a numerical point of view.

Remarks.

(1) It can be useful to remark that this procedure is not unique, that is, it produces as many o.n. sets in the lth Landau level (for any given shape of the lattice) as many d-MRA we are able to produce. This is not surprising because it once again reflects the infinite degeneracy of the Landau levels. Of course, among all the possibilities, we are interested in finding the one which minimizes the Coulomb energy.

(2) We observe that our strategy suggests some kind of transition related to the value of ν. The point is the following: in [3] our lattice was essentially constructed with a superposition of Gaussian functions, localized around different lattice sites. The same single-electron wavefunction was to be used for any value of the filling factor, the only difference being that the different wavefunctions were more or less distant depending on ν, which, as in the classical case, appears essentially as a parameter fixing the distance between two neighbouring electrons. Here the situation is more delicate. In fact, even if the expression for T_d is formally always the same, independent of the filling, see (4.1), the explicit expression changes because the coefficients h_n for a, say, 2-MRA are in general different from those of a 3-MRA or, in general, of any other d-MRA. We will show this in section 6, considering as an example the Haar 3-MRA. The consequence of this fact is that a d-MRA produces an N-electron wavefunction $\psi_d^{(N)}(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N)$ and two such functions, $\psi_{\frac{1}{3}}^{(N)}$ and $\psi_{\frac{1}{5}}^{(N)}$ for instance, need not have the same analytical dependence on their variables. A natural question to be considered is the following: is there any relation between a given $\psi_d^{(N)}(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N)$ and the plateaux of the Hall resistivity corresponding to $\nu = \frac{1}{d}$? We hope to consider this question in the near future.

As in [2], we now show just for completion how the procedure can be inverted, that is, how a set of coefficients satisfying equation (2.3) can be recovered by a function satisfying the ONC (4.2). The idea is quite simple: given such a function $K_d(s)$ we simply define

$$H_n = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} K_d(s)\, \mathrm{e}^{-\mathrm{i}ns\frac{2\pi}{a}}\, \mathrm{d}s \tag{4.5}$$

and it is now an easy computation to check that

$$\sum_{n\in\mathbb{Z}} H_n \overline{H}_{n+dl} = \delta_{l0} \qquad \forall l \in \mathbb{Z}. \tag{4.6}$$

Indeed, using formula $\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}nx\frac{2\pi}{a}} = a \sum_{n\in\mathbb{Z}} \delta(x - na)$ we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} H_n \overline{H}_{n+dl} &= \frac{1}{a} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \mathrm{d}s\, K_d(s)\, \mathrm{e}^{-\mathrm{i}ns\frac{2\pi}{a}} \int_{\mathbb{R}} \mathrm{d}s'\, \overline{K}_d(s')\, \mathrm{e}^{\mathrm{i}(n+dl)s'\frac{2\pi}{a}} \\ &= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \mathrm{d}s'\, K_d(s' - na) \overline{K}_d(s')\, \mathrm{e}^{\mathrm{i}dls'\frac{2\pi}{a}}.\end{aligned}$$

Our claim now follows from the assumptions on K_d.

Finally, it is not a big surprise that whenever we take $K_d(s)$ coincident with the function T_d in (4.1), then the coefficients H_n in (4.5) coincide with the h_n, $h_n = H_n$ for any n.

Before ending this section, it may be worth stressing that definitions (4.1) and (4.5), which are given here without any reasonable justification, are really a consequence of the analysis produced in [2], and, in particular, of the use of the kq-representation, which was crucial to make evident the relation between MRA and ONC.

5. The role of the kq-representation

In the previous section we have discussed the relation between a d-MRA and an o.n. set of single electron eigestates of the free Hamiltonian of a 2DEG in a given Landau level. Our main requirements are that we want to generate a triangular lattice and that we want our approach

to be useful for describing those values of the electron densities for which the plateaux are observed in experiments. These results, which extend in a relevant way the ones discussed in [2], have been obtained here without even mentioning the kq-representation. Nevertheless, as we have just remarked, this has proved to be a crucial guideline in [2]. For these reasons, and because the ONC can again be written in a much simpler form in the variables (k, q), we devote this section to restating our results by making use of the Zak transform. Moreover, this approach will also explain why the role of the geometry of the lattice and the particular Landau level are completely irrelevant as long as we are only interested in the orthonormality between the vectors ψ and $T_1^n T_2^m \psi$.

We begin with adapting the definitions sketched in section 2 to our situation, that is considering the unitary operators T_1 and T_2 defined in (3.19), $T_1 = \mathrm{e}^{\mathrm{i}Qa}$ and $T_2 = \mathrm{e}^{\mathrm{i}P\frac{2\pi}{a}}$. A simple extension of the original Zak's proof produces the following result:

Lemma 5.1. *The following set*

$$\psi_{kq}(p) = \sqrt{\frac{a}{2\pi}} \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}qna}\delta(p-k+na) \qquad k \in [0, a[\quad q \in \left[0, \frac{2\pi}{a}\right[\tag{5.1}$$

satisfies

$$T_1\psi_{kq}(p) = \mathrm{e}^{\mathrm{i}qa}\psi_{kq}(p) \qquad T_2\psi_{kq}(p) = \mathrm{e}^{\mathrm{i}k\frac{2\pi}{a}}\psi_{kq}(p) \tag{5.2}$$

and

$$\int_0^a \mathrm{d}k \int_0^{2\pi/a} \mathrm{d}q\, \psi_{kq}(p)\overline{\psi_{kq}(p')} = \delta(p-p'). \tag{5.3}$$

This means that any $\psi_{kq}(p)$ is a common eigenstate of T_1 and T_2 for all k and q and, moreover, that they form a complete set in $\mathcal{L}^2(\mathcal{D})$, $\mathcal{D} = [0, a[\times \left[0, \frac{2\pi}{a}\right[$, as in section 2.

The completeness of this set can be used to write $S_{n,m}$ in a different way. If we call $h(k, q)$ the Zak transform of $h(s)$,

$$h(k, q) = (Zh)(k, q) := \sqrt{\frac{2\pi}{a}} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}kna} h(q+na) \tag{5.4}$$

we can find that

$$S_{n,m} = \int_{\mathcal{D}} \mathrm{d}k\, \mathrm{d}q\, \mathrm{e}^{\mathrm{i}naq+\mathrm{i}km\frac{2\pi}{a}} |h(k, q)|^2 \tag{5.5}$$

which coincides essentially with the one in [2]. Incidentally, once again we have evidence of the canonicity of maps in the following sense: $\|\psi(x, y)\|_{\mathcal{L}^2(\mathbb{R}^2)} = \|h(s)\|_{\mathcal{L}^2(\mathbb{R})} = \|h(k, q)\|_{\mathcal{L}^2(\mathcal{D})}$. Requiring orthogonality only among ψ_{00} and $\psi_{n,dm}$, $n, m \in \mathbb{Z}$, we can repeat essentially the same steps as in [2] and we conclude that $S_{n,dm} = \delta_{n0}\delta_{m0}$ if and only if

$$J_d(k, q) := \sum_{l=0}^{d-1} \left| h\left(\frac{k+la}{d}, q\right)\right|^2 = \frac{d}{2\pi} \tag{5.6}$$

which looks more friendly than (3.24) because no integration appears!

The solution of this equation can be obtained starting from a set of coefficients $\{h_n\}$ arising from a d-MRA, and therefore satisfying (2.3). It is enough to define the function

$$t_d(k, q) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} h_n\, \mathrm{e}^{\mathrm{i}kn\frac{2\pi}{a}} \qquad (k, q) \in \mathcal{D}. \tag{5.7}$$

It is not hard to check now that (1) $t_d(k,q)$ satisfies condition (5.6), and (2) the inverse Zak transform of $t_d(k,q)$ returns T_d as in (4.1). Indeed we have

$$J_d(k,q) := \frac{1}{2\pi}\sum_{l,s\in\mathbb{Z}} h_s\overline{h_l}\,\mathrm{e}^{\mathrm{i}(s-l)\frac{2\pi k}{da}}\left(1+\mathrm{e}^{\mathrm{i}(s-l)\frac{2\pi}{d}}+\mathrm{e}^{\mathrm{i}(s-l)\frac{4\pi}{d}}+\cdots+\mathrm{e}^{\mathrm{i}(s-l)\frac{2(d-1)\pi}{d}}\right)$$
$$= \frac{1}{2\pi}\sum_{s,p\in\mathbb{Z}} h_s\overline{h_{s+p}}\,\mathrm{e}^{-\mathrm{i}p\frac{2\pi k}{da}}\left(\sum_{j=0}^{d-1}\left(\mathrm{e}^{-\mathrm{i}p\frac{2\pi}{d}}\right)^j\right).$$

Since $\sum_{j=0}^{d-1}\left(\mathrm{e}^{\mathrm{i}p\frac{2\pi}{d}}\right)^j$ is equal to d whenever $p=0,\pm d,\pm 2d,\pm 3d,\ldots$ and to 0 for all the other values of p, we conclude that, with a change of variable $m=\mathrm{d}p$,

$$J_d(k,q) := \frac{d}{2\pi}\sum_{s,m\in\mathbb{Z}} h_s\overline{h_{s+md}}\,\mathrm{e}^{-\mathrm{i}m\frac{2\pi k}{ad}}$$

which is finally equal to $\frac{d}{2\pi}$ as a simple consequence of equation (2.3).

As for our second claim, we have

$$(Z^{-1}t_d)(s) = \int_{\mathcal{D}}\psi_{kq}(p)t_d(k,q)\,\mathrm{d}k\,\mathrm{d}q$$
$$= \frac{\sqrt{a}}{2\pi}\sum_{n,m\in\mathbb{Z}} h_n\int_0^a \mathrm{d}k\int_0^{2\pi/a}\mathrm{d}q\,\mathrm{e}^{\mathrm{i}qma}\delta(p-k+ma)\,\mathrm{e}^{\mathrm{i}nk\frac{2\pi}{a}}$$
$$= \frac{1}{\sqrt{a}}\sum_{n\in\mathbb{Z}} h_n\int_0^a \mathrm{d}k\,\delta(p-k)\,\mathrm{e}^{\mathrm{i}nk\frac{2\pi}{a}}$$

which is equal to zero whenever $p\notin[0,a[$ while it gives a nontrivial contribution otherwise. Computing the integral for $p\in[0,a[$, our assertion easily follows.

In analogy with our results in the previous section and with those in [2], it is not hard to imagine how to use a function $t_d(k,q)$ satisfying the ONC in its form (5.5) to generate a set of coefficients H_n satisfying equation (2.3), reversing in this way the procedure and obtaining a d-MRA from an o.n. set in the LLL.

We want to conclude this section with a remark concerning the fact that the ONC looks identical for the square and for the triangular lattice and also for all the Landau levels different from the lowest one. These facts can be nicely understood in terms of the kq-representation. The idea is the following: let ψ be an eigenstate of the self-adjoint operator H_0 belonging to a certain eigenvalue ϵ, and let us suppose that H_0 commutes with the two unitary operators T_1 and T_2 as defined in (3.19). It is clear that, putting $\psi_{nm}=T_1^nT_2^m\psi$, $H_0\Psi_{nm}=\epsilon\Psi_{nm}$ for all integers n and m. We now want to answer the following question: is it possible to fix ψ in such a way that

$$S_{m,n} := \langle\psi,\psi_{nm}\rangle = \delta_{n0}\delta_{m0} \tag{5.8}$$

holds?

The answer follows quite easily now from lemma 5.1. In fact, using the completeness of the set ψ_{kq} associated with the magnetic translations, we can write

$$S_{m,n} = \int_{\mathcal{D}}\mathrm{d}k\,\mathrm{d}q\,\langle\psi,\psi_{kq}\rangle\left\langle\psi_{kq},T_1^nT_2^m\psi\right\rangle$$

and we go back to (5.5), putting $h(k,q)=\langle\psi_{kq},\psi\rangle$, simply because each ψ_{kq} is an eigenstate of both T_1 and T_2. As we see, the role of the shape of the lattice, as well as the particular Landau level we want to consider, is completely irrelevant for this result as we have explicitly seen in section 3.

6. An example: Haar for $\nu = \frac{1}{3}$

In this section we will discuss the easiest nontrivial example of a 'trial' ground state for the 2DEG arising from a 3-MRA with Haar-like behaviour. Let us first introduce the 3-MRA. This can be done as follows: the characteristic function of the unit interval $[0, 1[$, $H(x)$, is known to be a scaling function for a 2-MRA with coefficients $h_0 = h_1 = \frac{1}{\sqrt{2}}$, $h_i = 0$ for all the other values of i. The same function can be easily shown to be a scaling function also of a 3-MRA with coefficients $h_0 = h_1 = h_2 = \frac{1}{\sqrt{3}}$, $h_i = 0$ for all the other values of i. It is also obvious that equation (2.3) is verified. Therefore, we use this set of coefficients to construct our single electron wavefunction as shown before: the function $T_3(s)$ must be defined as

$$T_3(s) = \begin{cases} \frac{1}{\sqrt{3a}}\left(1 + \mathrm{e}^{\mathrm{i}s\frac{2\pi}{a}} + \mathrm{e}^{\mathrm{i}s\frac{4\pi}{a}}\right) & s \in [0, a[\\ 0 & \text{otherwise.} \end{cases}$$

Using T_3 in (3.11), together with the usual definition of the error function [12], we obtain

$$\psi_3(\underline{r}) = \frac{\mathrm{e}^{\frac{\mathrm{i}}{2}(xy - y^2/\sqrt{3})}}{\sqrt{6a\beta\left(1 + \frac{\mathrm{i}}{\sqrt{3}}\right)}2\pi^{1/4}}\left(G(x, y, y) + G\left(x, y, y + \frac{2\pi}{a}\right) + G\left(x, y, y + \frac{4\pi}{a}\right)\right) \tag{6.1}$$

where, as before, we have defined $\beta = \frac{3}{8}\left(1 - \frac{\mathrm{i}}{\sqrt{3}}\right)$ and

$$G(x, y, \alpha) = \mathrm{e}^{-\frac{\alpha^2}{4\beta} - \mathrm{i}\alpha(x - y/\sqrt{3})}\left\{\Phi\left(\frac{2\eta(x - y/\sqrt{3}) - \mathrm{i}\alpha + 2a\beta}{2\sqrt{\beta}}\right) - \Phi\left(\frac{2\beta(x - y/\sqrt{3}) - \mathrm{i}\alpha}{2\sqrt{\beta}}\right)\right\}. \tag{6.2}$$

$\psi_3(\underline{r})$ is now used to define $(\psi_3)_{nm}(\underline{r})$ as in (3.20), and one can explicitly check that

$$\int_{\mathbb{R}^2} \overline{\psi_3(\underline{r})}(\psi_3)_{n3m}(\underline{r})\,\mathrm{d}^2\underline{r} = \delta_{n0}\delta_{m0}.$$

An interesting feature of these functions is their asymptotic behaviour which can be found by considering the asymptotic behaviour of the error function [12]. After some algebra we obtain

$$\begin{aligned}\psi_3(\underline{r}) \simeq \frac{\sqrt{\beta}\,\mathrm{e}^{\frac{\mathrm{i}y}{2}(x - y/\sqrt{3})}}{\sqrt{3a}\pi^{3/4}}\,\mathrm{e}^{-\beta(x - y/\sqrt{3})^2}\Bigg\{&\frac{1}{2\beta(x - y/\sqrt{3}) - \mathrm{i}y} \\ &+ \frac{1}{2\beta(x - y/\sqrt{3}) - \mathrm{i}(y + 2\pi/a)} + \frac{1}{2\beta(x - y/\sqrt{3}) - \mathrm{i}(y + 4\pi/a)} \\ &- \mathrm{e}^{-\beta a^2 - a(2\beta(x - y/\sqrt{3}) - \mathrm{i}y)}\Bigg[\frac{1}{2\beta(x - y/\sqrt{3}) - \mathrm{i}y + 2a\beta} \\ &+ \frac{1}{2\beta(x - y/\sqrt{3}) - \mathrm{i}(y + 2\pi/a) + 2a\beta} \\ &+ \frac{1}{2\beta(x - y/\sqrt{3}) - \mathrm{i}(y + 4\pi/a) + 2a\beta}\Bigg]\Bigg\}.\end{aligned} \tag{6.3}$$

What is interesting about this formula is that it shows that whenever $x - y/\sqrt{3} \neq 0$ then $\psi_3(\underline{r})$ decreases very fast while, if $x - y/\sqrt{3} = 0$, then the decay is rather slow (like $1/|y|$ and independent of x). This is rather appealing because it suggests that, even if along a direction

the decay is the one expected by the Balian Low theorem [1], outside this set of zero measure the wavefunction decreases very fast *in both variables*.

Remark. In [2] we have discussed an example for $\nu = \frac{1}{2}$, also related to the Haar basis. We give here the result of that analysis in order to compare old and new results and, at the same time, to correct two typos. The wavefunction generating the o.n. set is

$$T_2(x, y) = \frac{\sqrt{a}\,\mathrm{e}^{-\mathrm{i}xy/2-y^2/2}}{4\pi^{3/4}}\left(\phi\left(\frac{x+a-\mathrm{i}y}{\sqrt{2}}\right) - \phi\left(\frac{x-\mathrm{i}y)}{\sqrt{2}}\right)\right.$$
$$\left. + \mathrm{e}^{-a^2/2+ay+\mathrm{i}xa}\left(\phi\left(\frac{x+a-\mathrm{i}(y-a)}{\sqrt{2}}\right) - \phi\left(\frac{x-\mathrm{i}(y-a)}{\sqrt{2}}\right)\right)\right)$$

whose asymptotic behaviour can be found with the help of [12]:

$$T_2(x, y) \simeq \frac{\sqrt{a}\,\mathrm{e}^{+\mathrm{i}xy/2-x^2/2}}{4\pi^{5/4}}\left(\frac{1}{x-\mathrm{i}y} + \frac{1}{x-\mathrm{i}(y-a)}\right.$$
$$\left. - \mathrm{e}^{-\pi-a(x-\mathrm{i}y)}\left(\frac{1}{x+a-\mathrm{i}y)} + \frac{1}{x+a-\mathrm{i}(y-a)}\right)\right).$$

In particular, we see that the decay of this function is rather slow, but for a set of zero measure, when compared with that of ψ_3.

Let us now use this wavefunction in the computation of the Coulomb energy. We repeat the same steps as in [3]. We want to compute the limit $\lim_{N\Rightarrow\infty} \frac{\langle H^{(N)}\rangle_{\psi^{(N)}}}{N}$, where $\psi^{(N)}$ is constructed as in (3.21). We write this mean value as a sum of two terms, a *kinetic* contribution coming from H_0, which is very well known since all the single-electron wavefunctions are eigenstates of H_0, and a Coulomb correction E_c which is the one we really need to compute. Therefore

$$\lim_{N\to\infty} \frac{\langle H^{(N)}\rangle_{\psi^{(N)}}}{N} = \frac{\hbar\omega}{2} + E_c \qquad E_c = E_W + \delta E \tag{6.4}$$

where E_W is the classical energy per electron of the Wigner crystal, $E_W = -0.7821\sqrt{\nu}$ [3], while δE is the quantum correction,

$$\delta E = \lim_{N\to\infty} \frac{1}{2} \sum_{(n,m)\in\mathcal{C}} \left\{[E_d((n,m),0) - E_{ex}((n,m),0)] - \frac{1}{|\underline{R}_{n,m}|}\right\}. \tag{6.5}$$

Here $\mathcal{C}$ is the subset of $\mathbb{Z}\times\mathbb{Z}$ corresponding to the sub-lattice we are considering. In this case, for instance, $\mathcal{C} = (\mathbb{Z}, 3\mathbb{Z})\backslash(0,0)$ to avoid self-energy divergences. Moreover, as in section 3, we have $\underline{R}_{n,m} = (X_{n,m}, Y_{n,m})$, while the direct and the exchange contributions are

$$E_d((n,m),0) := \int \mathrm{d}^2\underline{r}_1\,\mathrm{d}^2\underline{r}_2 \frac{|\psi_{n,m}(\underline{r}_1)|^2|\psi(\underline{r}_2)|^2}{|\underline{r}_1-\underline{r}_2|} \tag{6.6}$$

and

$$E_{ex}((n,m),0) := \int \mathrm{d}^2\underline{r}_1\,\mathrm{d}^2\underline{r}_2 \frac{\overline{\psi_{n,m}(\underline{r}_1)\psi(\underline{r}_2)}\psi(\underline{r}_1)\psi_{n,m}(\underline{r}_2)}{|\underline{r}_1-\underline{r}_2|}. \tag{6.7}$$

In this paper, we are not interested in giving a detailed result for δE, since the CPU time required to compute each integral is very high (more than 1 day for each integration with a reasonable approximation). For this reason, we only compute here the integrals using a Monte Carlo method with 250 000 points and restricting the integration to a large but compact region

(taking advantage of the decay behaviour of $\psi_{n,m}$ and checking first the normalization of each wavefunction). With this very rough approximation the CPU time for each integration is about 4 h. Moreover, we disregard the contributions coming from the exchange integrals, which are usually some orders of magnitude smaller than the direct one [3]. Finally, due to the fact that the direct energies $E_d((n, m), 0)$ go to zero when (n, m) increase, we also work with a finite (but big) lattice.

For the above reasons we do not expect that the present result could have any real meaning, but nevertheless it is useful because it gives a first idea of the numbers appearing in the game. The result we have obtained is $\delta E = 0.3184$ which must be compared with the much lower result in [3], 0.0657. It is useful to stress that this does not imply at all that the Haar wavefunction is physically useless, but only that a different numerical integration technique has to be used. A more detailed numerical analysis of the procedure is now being performed. This will include the computation of the Coulomb energy for different values of the filling factor as well as the use of different d-MRA. We hope to produce physically relevant results in the near future.

7. Outcome

In this paper we have carried on the analysis of the connection between a MRA of $\mathcal{L}^2(\mathbb{R})$ and the FQHE, first proposed in [2]. In particular, we have shown how a single electron wavefunction which, together with its magnetic translates, produces an o.n. set in the LLL can be constructed starting from a d-MRA. This procedure works for $\nu = \frac{1}{d}$, $d \in \mathbb{N}$ and $d \geqslant 2$. We have also shown that this procedure can be essentially inverted since to any o.n. basis of translated functions of the LLL (corresponding $\nu = \frac{1}{d}$) corresponds a set of coefficients satisfying the main condition of a d-MRA of $\mathcal{L}^2(\mathbb{R})$.

All of these results have been obtained working with a triangular lattice, which is the one suggested by classical and quantum energetic considerations. The extension to higher Landau levels has also been briefly outlined.

We have also given an example of our construction for $\nu = \frac{1}{3}$, using the Haar 3-MRA, and for the related o.n. basis in the configuration space we have also obtained an approximated value of the Coulomb energy.

It is clear that our method produces a wide set of possible o.n. functions in the LLL, and our future interest consists mainly in going beyond the simple Haar choice trying to minimize the Coulomb energy so as to recover completely the experimental data.

As already mentioned, it will be very interesting to also analyse the (possible) connection between the different wavefunctions $\psi_d^{(N)}$ produced by our method and the plateaux corresponding to $\nu = \frac{1}{d}$. Of course, this mechanism is completely *non-standard*, so that an analysis of the relations between our approach and the one originated by Laughlin should also be worked out.

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